ON 2-SYLOW INTERSECTIONS

BY

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ABSTRACT

Let z be an involution in the finite group G and suppose that z belongs to the center of a Sylow subgroup of G. If z belongs to a unique Sylow subgroup of G and if G is not a trivial intersection group, then G is not a simple group.

Let G be a finite group and let g be a p-element of G. We will say that g is a *central p-element* if it belongs to a center of a Sylow p-subgroup of G. If g belongs to a unique Sylow p-subgroup of G then it will be called a *concealed p-element.*

The aim of this note is to prove the following theorem, the proof of which depends crucially on a recent theorem of Shult [1, p. 62].

THEOREM A. *Let G be a finite group and suppose that G contains a central concealed involution z. Then $\langle z^G \rangle = N$, the normal closure of z in G, has a center $Z(N)$ of odd order and*

$$N/Z(N) \cong N_1 \times \cdots \times N_r \times M$$

where M has an elementary abelian Sylow 2-subgroup and a normal 2-complement and each N_i is isomorphic to $PSL(2,2^{n_i})$, $Sz(2^{n_i})$ or $PSU(3,2^{n_i})$ for some n_i.

A subgroup D of a finite group G is called a 2-*Sylow intersection* if there exist distinct Sylow 2-subgroups S_1 and S_2 of G such that $D = S_1 \cap S_2$. The group G is called a *TI-group* if all its 2-Sylow intersections are trivial.

Since the simple groups mentioned in Theorem A are *TI*-groups, we get the following simplicity criterion.

THEOREM B. *Let G be a finite group and suppose that G contains a central concealed involution. If G is not a TI-group then G is not a simple group.*

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Finally we get the following generalization of theorem 1 in Suzuki's paper [2]:

THEOREM C. *Let G be a non-abelian simple finite group and suppose that a central involution of G belongs to no 2-Sylow intersection. Then G is isomorphic to one of the groups $PSL(2,q)$, $Sz(q)$ or $PSU(3,q)$ for some $q = 2^n > 2$.*

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PROOF OF THEOREM A. It is obvious that any conjugate of z belongs to a unique Sylow 2-subgroup of G and hence to its center. Let S be the Sylow 2-subgroup of G containing z.

LEMMA 1. *Let $T = \langle z^g \mid g \in G,\ z^g \in C_G(z)\rangle$. Then $T \subseteq Z(S)$.*

PROOF. Let $g \in G$ such that $z^g \in C_G(z)$. Since $S \lhd C_G(z)$, it follows that $z^g \in S$, hence $z \in S^{g^{-1}} \cap S$. Thus $g \in N_G(S)$ and $z^g \in Z(S)$, yielding $T \subseteq Z(S)$.

LEMMA 2. $N(S) \subseteq N(T)$.

PROOF. Let $n \in N(S)$; if $z^g \in C_G(z)$ then by Lemma 1

$$z^{gn} \in Z(S) \subseteq C_G(z), \text{ hence } T^n \subseteq T.$$

LEMMA 3. *$N_G(T) \cap T^g \subseteq T$ for all $g \in G$.*

PROOF. Suppose that $x \in T$ and $x^g \in N(T) \cap T^g$. Let D be the conjugate class of G containing z. Since by Lemma 1 $T \subseteq Z(S)$, $|T \cap D|$ is odd and as $x^g \in N(T)$ is an involution, x^g centralizes an element z^n of $T \cap D$, where $n \in N(S) \subseteq N(T)$. Thus $z^{ng^{-1}}$ centralizes x and as $x \in T$, also z centralizes x and $S \subseteq C_G(x)$. Obviously there exists $c \in C_G(x)$ such that $z^{ng^{-1}c} \in S$. But then $z^{ng^{-1}c} \in T \cap T^{ng^{-1}c} = T \cap T^{g^{-1}c}$ and consequently $g^{-1}c \in N(S) \subseteq N(T)$, hence $c^{-1}g \in N(T)$. It follows that $x^g = x^{c^{-1}g} \in T$, as required.

Theorem A follows immediately from Lemma 3 and the fusion theorem of Shult [1, p. 62].

REFERENCES

1. G. Glauberman, *Global and local properties of finite groups*, Finite Simple Groups, Academic Press, 1971.

2. M. Suzuki, *Finite groups of even order in which Sylow 2-groups are independent*, Ann. Math. **80** (1964), 58–77.

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